Kerr Trisector Closure:
A Cross-Domain Self-Consistency Test for General Relativity

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Many tests of General Relativity assess its predictions within a single observational channel. We introduce the *Kerr Trisector Closure* (KTC), a cross-domain consistency test that compares independent inferences of the mass and spin of a Kerr black hole from inspiral gravitational-wave dynamics, quasinormal-mode ringdown spectroscopy, and horizon-scale imaging. The test is summarized by a Hotelling-type closure statistic[1] T^2, constructed from the sectoral parameter estimates and their covariances, with a null distribution defined by the hypothesis of a common Kerr parameter pair. The consistency among the three estimates corresponds to T^2 values compatible with the null expectation, while an excess quantifies the level of inconsistency and can be decomposed to identify the sector(s) driving the tension, without specifying an alternative theory.

I. INTRODUCTION

General Relativity (GR) has withstood a century of increasingly precise tests[2, 3]. Its predictions describe the motion of planets, the timing of satellite clocks, and the gravitational waves detected by ground- and space-based observatories[3, 4]. From Eddington's 1919 eclipse expedition to the first gravitational-wave detections by LIGO and Virgo, observations have supported Einstein's geometric description of gravity[5, 6]. Despite this success, many tests probe individual regimes in isolation. Most validations focus on a single sector, such as weak-field lensing, post-Newtonian corrections, wave propagation, or near-horizon phenomena, rather than enforcing consistency of a single metric description across all regimes[3, 7].

In GR, a common spacetime metric determines the dynamics of massive bodies, the response of the horizon to perturbations, and the propagation of photons. A cross-domain test therefore requires that parameters inferred from mechanically, radiatively, and optically defined observables agree within their experimental uncertainties.

The Kerr solution provides a concrete setting for such a requirement[8, 9]. The Einstein field equations yield a family of stationary, axisymmetric vacuum spacetimes parameterized by the mass M and the Kerr spin parameter a, defined by $a = J/M$ for an uncharged rotating black hole (geometric units $G = c = 1$).[9] (It is often reported instead via the dimensionless spin $\chi \equiv a/M = J/M^2$.) Observable properties of geodesics, redshift, precession, quasinormal-mode spectra, and shadow morphology depend on (M, a)[9–11]. In GR, each observable sector should infer statistically consistent values of (M, a)[12]. However, existing analyses typically treat dynamical, ringdown, and imaging measurements independently and do not impose a combined consistency requirement among the resulting parameter estimates[7, 13]. A discrepancy among sectoral estimates (e.g., between gravitational-wave-inferred mass and the mass implied by shadow size, or between spin inferred from quasinormal modes and from orbital dynamics) would indicate tension with the Kerr description and can be quantified statistically.

The objective of this study is to establish a formal framework for a self-consistency test, referred to as the Kerr Trisector Closure (KTC). The KTC combines three observational domains for the same astrophysical black hole: inspiral gravitational-wave dynamics, ringdown quasinormal-mode spectroscopy, and horizon-scale imaging through photon-ring geometry and shadow measurements. Under GR, a single pair (M, a) must be consistent with all three domains within their joint uncertainties. The formulation of this closure principle includes the definition of independent sector likelihoods, the derivation of a closure statistic with a defined null distribution, and the identification of observational regimes that can implement the test in practice. The framework evaluates whether independent determinations of (M, a) inferred from different physical processes are mutually consistent.

II. KERR GEOMETRY AND OBSERVATIONAL SECTORS

The external spacetime of an uncharged, stationary, and axisymmetric rotating black hole is uniquely described, within General Relativity, by the Kerr solution[8, 9]. In Boyer–Lindquist coordinates (t, r, θ, ϕ), and working in geometric units $G = c = 1$, the line element takes the canonical form

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$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}\, dt\, d\phi + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2, \tag{1}$$

where the metric functions are defined as

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta, \tag{2}$$

$$\Delta(r) = r^2 - 2Mr + a^2. \tag{3}$$

Here M denotes the total mass of the black hole, and $a = J/M$ is the Kerr spin parameter, where J is the total angular momentum (geometric units $G = c = 1$). Where a dimensionless spin is required we use $\chi \equiv a/M = J/M^2$. Throughout this work we employ the metric signature $(-+++)$.

The event horizons are located at the roots of $\Delta = 0$, that is

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{4}$$

with r_+ corresponding to the outer (event) horizon and r_- to the inner (Cauchy) horizon. In the limit $a \to 0$, Eq. (1) reduces to the Schwarzschild metric, whereas the extremal case $a \to M$ corresponds to a maximally rotating black hole. All astrophysically observable quantities derived from stationary black holes, whether related to dynamics, perturbations, or light propagation, can be expressed as functionals of the two parameters M and a.

Dynamical Sector. The dynamical or *orbital* sector concerns the motion of massive particles and binary systems in the Kerr spacetime. In the weak-field, slow-velocity limit, the equations of motion reduce to the parameterized post-Newtonian (PPN) expansion, but as the separation decreases the full geodesic dynamics must be used. The equations governing timelike geodesics $x^\mu(\tau)$ follow from the Lagrangian

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1, \tag{5}$$

leading to four first integrals associated with the stationarity and axisymmetry of the metric: the specific energy E, the specific angular momentum L_z, the Carter constant Q, and the normalization condition. Here Q denotes the Carter constant (sometimes written $\mathcal{Q}$), defined by separation of the Hamilton–Jacobi equation; for timelike geodesics it may be written as[14]

$$Q \equiv p_\theta^2 + \cos^2\theta\left[a^2(1-E^2) + \frac{L_z^2}{\sin^2\theta}\right], \tag{6}$$

and it vanishes for equatorial motion. The system of first-order equations reads

$$\Sigma\frac{dr}{d\tau} = \pm\sqrt{R(r)}, \tag{7}$$

$$\Sigma\frac{d\theta}{d\tau} = \pm\sqrt{\Theta(\theta)}, \tag{8}$$

$$\Sigma\frac{d\phi}{d\tau} = \frac{L_z}{\sin^2\theta} - aE + \frac{aP(r)}{\Delta}, \tag{9}$$

$$\Sigma\frac{dt}{d\tau} = a(L_z - aE\sin^2\theta) + \frac{(r^2+a^2)P(r)}{\Delta}, \tag{10}$$

where

$$R(r) = [P(r)]^2 - \Delta\big[r^2 + (L_z - aE)^2 + Q\big], \tag{11}$$

$$\Theta(\theta) = Q - \cos^2\theta\left[a^2(1-E^2) + \frac{L_z^2}{\sin^2\theta}\right], \tag{12}$$

$$P(r) = E(r^2 + a^2) - aL_z. \tag{13}$$

These equations, when integrated numerically or perturbatively, describe precession effects, orbital resonances, and the inspiral dynamics leading to gravitational-wave emission. Thus, the dynamical sector constrains M and a through the motion of massive bodies and the phasing of gravitational waves during inspiral.

Perturbative Sector. Following the merger of a binary system or a perturbation of an isolated Kerr black hole, the geometry settles back to equilibrium through the emission of gravitational radiation.[15, 16] This process is governed by the linearized Einstein equations on a Kerr background, which can be recast (in the Newman–Penrose formalism) as the Teukolsky master equation for the outgoing Weyl scalar ψ_4 (spin weight $s = -2$).[17]

$$\mathcal{T}[\psi_4] = 0, \tag{14}$$

where $\mathcal{T}$ is a second-order differential operator depending on M and a. Assuming separability in spin-weighted spheroidal harmonics and imposing quasinormal-mode boundary conditions (purely ingoing at the horizon and purely outgoing at infinity) yields a discrete spectrum of complex frequencies $\omega_{lmn}(M,a)$, whose real part gives the oscillation frequency and whose imaginary part determines the damping rate[10, 17, 18]. Explicitly,

$$\omega_{lmn} = \omega^{(r)}_{lmn}(M,a) + i\,\omega^{(i)}_{lmn}(M,a), \qquad \tau_{lmn} = \frac{1}{|\omega^{(i)}_{lmn}|}. \tag{15}$$

These frequencies provide an independent probe of the spacetime geometry near the horizon. In the framework of the Kerr Trisector Closure, the ringdown signal constrains M and a purely through perturbative dynamics, without reference to orbital motion or light propagation.

Geometrical–Optical Sector. The third sector pertains to null geodesics and the propagation of light in the Kerr spacetime. For photons, the Hamilton–Jacobi equation separates in the same manner as for massive particles, with the null condition replacing the timelike normalization (equivalently, setting the rest mass to zero). In the separated Hamilton–Jacobi formulation, null geodesics are characterized by the conserved ratios

$$\xi \equiv \frac{L_z}{E}, \qquad \eta \equiv \frac{Q}{E^2}, \tag{16}$$

where E is the photon energy at infinity, L_z the axial angular momentum, and Q the Carter constant as defined above (not an electric charge). Equivalently, one may introduce the separation constant K with $K \equiv Q + (L_z - aE)^2$; the shadow formulas below are written in terms of (ξ, η). The boundary of the Kerr shadow as seen by a distant observer at inclination i is most conveniently described by the celestial coordinates (α, β) on the observer's image plane,[19]

$$\alpha = -\frac{\xi}{\sin i}, \qquad \beta = \pm\sqrt{\eta + a^2\cos^2 i - \xi^2\cot^2 i}, \tag{17}$$

which map the critical photon parameters (ξ, η) into observable angular impact coordinates. On the shadow boundary these parameters are generated by unstable spherical photon orbits of radius r, for which

$$\xi(r) = \frac{r^2(r-3M) + a^2(r+M)}{a(M-r)},$$
$$\eta(r) = \frac{r^3\big(4a^2M - r(r-3M)^2\big)}{a^2(M-r)^2}. \tag{18}$$

A convenient one-number summary of the shadow size is the horizontal diameter in the image plane,

$$d_{\rm sh}(M,a,i) \equiv \alpha_{\max}(M,a,i) - \alpha_{\min}(M,a,i), \tag{19}$$

with the corresponding angular diameter given by $\theta_{\rm sh} = d_{\rm sh}/D$ for a source at distance D. Very-long-baseline interferometry (VLBI) measurements of black-hole shadows, such as those from the Event Horizon Telescope, thus yield an optical constraint on (M, a) independent of dynamical or perturbative data[13, 20–22].

III. FORMULATION OF THE KERR TRISECTOR CLOSURE

Let the parameters of the Kerr spacetime inferred from the three independent observational sectors be denoted collectively as

$$\widehat{\Theta}_{\rm insp} = (\widehat{M}_{\rm insp}, \, \widehat{a}_{\rm insp}),$$
$$\widehat{\Theta}_{\rm ring} = (\widehat{M}_{\rm ring}, \, \widehat{a}_{\rm ring}),$$
$$\widehat{\Theta}_{\rm img} = (\widehat{M}_{\rm img}, \, \widehat{a}_{\rm img}), \tag{20}$$

each accompanied by an empirical covariance matrix $\Sigma_k \in \mathbb{R}^{2\times 2}$, where $k \in \{\text{insp}, \text{ring}, \text{img}\}$. Under General Relativity, all three estimates must represent statistically consistent measurements of a single pair $\Theta = (M, a)$.

A. Sector-Specific Likelihoods

Each observational sector produces a likelihood function $\mathcal{L}_k(\Theta)$ describing the probability of the measured data d_k given the theoretical prediction for a Kerr black hole with parameters Θ. For the inspiral gravitational-wave data, recorded across a network of detectors, the likelihood under the Gaussian-noise approximation is expressed as

$$\mathcal{L}_{\rm insp}(\Theta) \propto \exp\left[-\tfrac{1}{2}\langle d_{\rm insp} - h_{\rm GR}(\Theta) \,|\, d_{\rm insp} - h_{\rm GR}(\Theta)\rangle\right], \tag{21}$$

where $h_{\rm GR}(\Theta)$ is the waveform predicted by GR and the inner product is defined by

$$\langle x|y\rangle = 4\,\Re\int_0^\infty \frac{\tilde{x}(f)\tilde{y}^*(f)}{S_n(f)}\,df, \tag{22}$$

with $S_n(f)$ denoting the one-sided detector noise spectral density. Parameter estimation in this sector yields a posterior distribution $p(\Theta|d_{\rm insp}) \propto \mathcal{L}_{\rm insp}(\Theta)\pi(\Theta)$, from which the mean estimate $\widehat{\Theta}_{\rm insp}$ and covariance $\Sigma_{\rm insp}$ are obtained.

The post-merger ringdown is modeled as a superposition of damped quasinormal modes of the Kerr spacetime,

$$h(t) = \sum_{lmn} A_{lmn} \exp\left[-\frac{t}{\tau_{lmn}(\Theta)}\right] \cos[\omega_{lmn}(\Theta)\,t + \phi_{lmn}], \tag{23}$$

where each mode is labeled by angular indices (l, m, n), complex frequency $\omega_{lmn} = \omega^{(r)}_{lmn} + i\omega^{(i)}_{lmn}$, and damping time $\tau_{lmn} = 1/|\omega^{(i)}_{lmn}|$. Given measured frequencies and damping rates $(\widehat{\omega}_{lmn}, \widehat{\tau}_{lmn})$, the likelihood for the ringdown sector is taken as

$$\mathcal{L}_{\rm ring}(\Theta) \propto \exp\left[-\frac{1}{2}\sum_{lmn}\left\{\frac{[\widehat{\omega}^{(r)}_{lmn} - \omega^{(r)}_{lmn}(\Theta)]^2}{\sigma^2_{\omega,lmn}} + \frac{[\widehat{\tau}_{lmn} - \tau_{lmn}(\Theta)]^2}{\sigma^2_{\tau,lmn}}\right\}\right], \tag{24}$$

from which the corresponding posterior moments yield $\widehat{\Theta}_{\rm ring}$ and $\Sigma_{\rm ring}$. The imaging sector constrains the spacetime parameters through null geodesic structure. In the standard asymptotic-observer description, the Kerr shadow is the set of critical photon directions (α, β) in Eq. (17) generated by unstable spherical photon orbits through Eq. (18). From this curve one may define a shadow diameter proxy

$$d_{\rm sh}(\Theta, i) \equiv \alpha_{\max}(\Theta, i) - \alpha_{\min}(\Theta, i), \tag{25}$$

and the corresponding angular diameter $\theta_{\rm sh}(\Theta, i, D) = d_{\rm sh}(\Theta, i)/D$.The imaging likelihood, based on an observed angular shadow diameter $\widehat{\theta}_{\rm sh}$ (or an equivalent size estimator extracted from the reconstructed brightness distribution), is written as

$$\mathcal{L}_{\rm img}(\Theta) \propto \exp\left(-\frac{1}{2} \frac{[\widehat{\theta}_{\rm sh} - \theta_{\rm sh}(\Theta, i, D)]^2}{\sigma_\theta^2} \right). \tag{26}$$

In practice, $d_{\rm sh}(M, a, i)$ is evaluated numerically by parameterizing the shadow boundary with the spherical photon-orbit radius r, computing $(\xi(r), \eta(r))$ from Eq. (18), mapping to (α, β) via Eq. (17), and then extracting $\alpha_{\min}$ and $\alpha_{\max}$ over the admissible range of r.

B. Gaussian (Laplace) Approximation

The closure statistic is written in terms of sectoral point estimates $\widehat{\Theta}_k$ and covariances Σ_k. Throughout we adopt the standard Gaussian (Laplace) approximation in which each sector posterior is expanded to second order about its maximum (or mean),

$$p_k(\Theta \mid d_k) \approx \mathcal{N}(\widehat{\Theta}_k, \Sigma_k), \tag{27}$$

with Σ_k given by the inverse Fisher information (or the local curvature of $-\ln p_k$). This approximation is justified asymptotically when the likelihood is sufficiently informative (e.g., high SNR and weak prior influence) and the posterior is unimodal. It can fail in regimes with strong parameter degeneracies, multimodality, pronounced non-Gaussian tails, or hard physical boundaries (e.g., near-extremal spin or low-SNR ringdown), in which case T^2 should be calibrated by Monte Carlo injection studies using the full posteriors.

Marginalization over nuisance parameters i and D yields the joint estimate $\widehat{\Theta}_{\rm img}$ and covariance $\Sigma_{\rm img}$.

C. Global Parameter Estimation and Closure Residuals

The three independent estimates are assembled into a composite parameter vector

$$\widehat{\boldsymbol{\Theta}} = \begin{bmatrix} \widehat{\Theta}_{\rm insp} \\ \widehat{\Theta}_{\rm ring} \\ \widehat{\Theta}_{\rm img} \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} \Sigma_{\rm insp} & \Sigma_{\rm insp,ring} & \Sigma_{\rm insp,img} \\ \Sigma_{\rm ring,insp} & \Sigma_{\rm ring} & \Sigma_{\rm ring,img} \\ \Sigma_{\rm img,insp} & \Sigma_{\rm img,ring} & \Sigma_{\rm img} \end{bmatrix}, \tag{28}$$

where $\mathbf{C}$ is the full covariance matrix of the stacked estimator. If the three sectors are statistically independent (e.g., disjoint datasets and negligible shared systematics), then the cross-covariances vanish and $\mathbf{C}$ reduces to the block-diagonal form blockdiag$(\Sigma_{\rm insp}, \Sigma_{\rm ring}, \Sigma_{\rm img})$. If two sectors share data products, calibration, or common nuisance parameters (as in multiband GW inference, or imaging constrained by EM distance priors), the corresponding blocks $\Sigma_{k\ell}$ should be retained. It is convenient to write the model for a common Kerr parameter pair $\Theta \in \mathbb{R}^2$ in stacked form as

$$\widehat{\boldsymbol{\Theta}} = \mathbf{A}\,\Theta + \boldsymbol{\varepsilon}, \qquad \mathbf{A} \equiv \nVdash_3 \otimes \mathbb{I}_2, \qquad \mathrm{Cov}(\boldsymbol{\varepsilon}) = \mathbf{C}, \tag{29}$$

where $\mathbb{I}_2$ is the 2×2 identity. The generalized least-squares (GLS) estimator of the common mean is then

$$\bar{\Theta} = \left(\mathbf{A}^\top \mathbf{C}^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^\top \mathbf{C}^{-1} \widehat{\boldsymbol{\Theta}}, \tag{30}$$

which coincides with minimizing the quadratic form $\left(\widehat{\boldsymbol{\Theta}} - \mathbf{A}\Theta\right)^\top \mathbf{C}^{-1} \left(\widehat{\boldsymbol{\Theta}} - \mathbf{A}\Theta\right)$.

The residual must be defined in the $\mathbf{C}^{-1}$ inner product to be orthogonal to the fitted mean. Define the (GLS) projection matrix onto the orthogonal complement of col$(\mathbf{A})$ as

$$\mathbf{P} \equiv \mathbb{I}_6 - \mathbf{A}\left(\mathbf{A}^\top \mathbf{C}^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^\top \mathbf{C}^{-1}. \tag{31}$$

The closure residual vector is then

$$\mathbf{r} \equiv \mathbf{P}\,\widehat{\boldsymbol{\Theta}}, \tag{32}$$

which satisfies $\mathbf{A}^\top \mathbf{C}^{-1} \mathbf{r} = 0$ by construction.

D. Definition of the Kerr Trisector Statistic

The degree of self-consistency among the three sectors is quantified by the Kerr Trisector statistic

$$T^2 = \mathbf{r}^\top \mathbf{C}^{-1} \mathbf{r} = \widehat{\boldsymbol{\Theta}}^\top \mathbf{P}^\top \mathbf{C}^{-1} \mathbf{P}\,\widehat{\boldsymbol{\Theta}}. \tag{33}$$

Under the null hypothesis that all sectors are governed by a common set of Kerr parameters Θ_* consistent with GR, the statistic T^2 reduces to a generalized least-squares (GLS) residual sum of squares. If, under the Gaussian (Laplace) approximation, the stacked estimator is well described by $\widehat{\boldsymbol{\Theta}} \sim \mathcal{N}(\mathbf{A}\Theta_*, \mathbf{C})$ with $\mathbf{C}$ treated as known and nonsingular, then

$$T^2 \sim \chi^2_\nu, \qquad \nu = \mathrm{rank}(\mathbf{P}) = 6 - \mathrm{rank}(\mathbf{A}) = 6 - 2 = 4. \tag{34}$$

Equivalently, for K sectors and a p-dimensional parameter vector, $\nu = (K-1)p$ provided $\mathbf{A}$ has full column rank. If $\mathbf{C}$ is estimated from data (or if the sector posteriors are only approximately Gaussian), the chi-squared law should be interpreted as asymptotic.

Deviations from GR are identified whenever

$$T^2 > \chi^2_{1-\alpha}(\nu), \tag{35}$$

for a chosen significance level α (typically $\alpha = 0.05$). If this criterion is satisfied, the null hypothesis of a single consistent Kerr metric is rejected, implying a breakdown of metric universality across the dynamical, perturbative, and optical domains.

In the language of multivariate statistics, T^2 serves as a Hotelling-type measure of distance between the sectoral parameter estimates. Its value encapsulates not merely the discrepancy among the means but also the combined covariance structure of the underlying posteriors. A small T^2 indicates internal closure of GR's predictions, whereas a significant excess would constitute direct evidence that the Kerr solution fails to account for at least one class of observables.

IV. ANALYTICAL BEHAVIOUR AND SENSITIVITY

The statistical behaviour of the Kerr Trisector statistic can be understood by examining the geometry of the parameter posteriors in the two-dimensional (M, a) space. Each observational sector yields an approximate Gaussian posterior distribution

$$p_k(\Theta) \propto \exp\Big[-\tfrac{1}{2}(\Theta - \Theta_k^*)^\top \Sigma_k^{-1} (\Theta - \Theta_k^*) \Big], \tag{36}$$

where Θ_k^* denotes the sector-specific maximum-likelihood estimator and Σ_k the corresponding covariance matrix derived from the Fisher information of that dataset.

A. Posterior Geometry and Degeneracy Directions

In the vicinity of the true parameters $\Theta_* = (M_*, a_*)$, each covariance matrix Σ_k defines an error ellipse whose principal axes indicate the degeneracy directions of the measurement. For the inspiral sector, the likelihood surface is elongated along lines of constant *chirp mass* $\mathcal{M} = (M_1 M_2)^{3/5}(M_1 + M_2)^{-1/5}$, which dominates the phasing of the waveform. To first order, the inspiral constrains M through $\mathcal{M}$ and is weakly sensitive to a, giving rise to a degeneracy nearly parallel to the a-axis. In contrast, the ringdown sector probes the ratio a/M via the discrete quasinormal-mode spectrum, whose frequencies $\omega_{lmn}(M, a)$ approximately obey the scaling

$$\omega_{lmn}(M, a) \simeq \frac{1}{M}\, \tilde{\omega}_{lmn}(a/M), \tag{37}$$

where $\tilde{\omega}_{lmn}$ is a dimensionless function tabulated for each mode. This relation couples mass and spin in opposite senses compared with the inspiral case, resulting in a degeneracy direction almost orthogonal to that of the inspiral posterior. The imaging sector, governed by photon-ring geometry, constrains the combination M/D through the apparent angular diameter of the shadow. In the asymptotic-observer description of Sec. II, the (horizontal) shadow diameter in the image plane is

$$d_{\rm sh}(M, a, i) \equiv \alpha_{\max}(M, a, i) - \alpha_{\min}(M, a, i), \tag{38}$$

so that the corresponding angular diameter is $\theta_{\rm sh}(\Theta, i, D) = d_{\rm sh}(\Theta, i)/D$, where D is the source distance. For fixed D, the constraint is predominantly aligned with the mass axis. Because the three sectors probe complementary physical regimes (orbital dynamics, horizon perturbations, and null geodesics), their degeneracy directions intersect nearly orthogonally, producing a highly localized intersection region when GR holds.

B. Perturbative Expansion around the GR Solution

Let the true Kerr parameters predicted by GR be $\Theta_* = (M_*, a_*)$, and define for each sector a small deviation vector

$$\delta\Theta_k = \widehat{\Theta}_k - \Theta_*. \tag{39}$$

In the Gaussian (Laplace) approximation, the expectation value of the statistic T^2 defined in Eq. (33) can be written in terms of the projected residual $\mathbf{r} = \mathbf{P}\widehat{\boldsymbol{\Theta}}$. Let

$$\boldsymbol{\mu} \equiv \mathbb{E}[\widehat{\boldsymbol{\Theta}}], \qquad \mathbf{b} \equiv \boldsymbol{\mu} - \mathbf{A}\Theta_* \tag{40}$$

denote the mean stacked estimate and its systematic offset from the GR prediction $\mathbf{A}\Theta_*$. Then $\mathbb{E}[\mathbf{r}] = \mathbf{P}\boldsymbol{\mu} = \mathbf{P}\mathbf{b}$, and

$$\begin{aligned} \langle T^2 \rangle &= \mathrm{Tr}(\mathbf{P}) + \langle \mathbf{r} \rangle^\top \mathbf{C}^{-1} \langle \mathbf{r} \rangle \\ &= \nu + \mathbf{b}^\top \mathbf{P}^\top \mathbf{C}^{-1} \mathbf{P}\, \mathbf{b}, \end{aligned} \tag{41}$$

where $\nu \equiv \mathrm{rank}(\mathbf{P}) = (K-1)p$ is the number of residual degrees of freedom. Under perfect closure (no systematic offsets so $\mathbf{b} = 0$) this reduces to $\langle T^2 \rangle = \nu$, while any systematic displacement increases $\langle T^2 \rangle$ by the quadratic form $\mathbf{b}^\top \mathbf{P}^\top \mathbf{C}^{-1} \mathbf{P}\mathbf{b}$, which weights the mismatch by the full covariance structure.

C. Sensitivity and Scaling Behaviour

The sensitivity of the KTC to small deviations can be estimated through a Fisher-matrix analysis. Let $\mathcal{I}_k = \Sigma_k^{-1}$ denote the Fisher information matrix for sector k. A local Taylor expansion of each likelihood around Θ_* gives

$$-2 \ln \mathcal{L}_k(\Theta) \simeq (\Theta - \Theta_*)^\top \mathcal{I}_k (\Theta - \Theta_*), \tag{42}$$

so that the joint likelihood entering the closure test scales as

$$-2\ln\mathcal{L}_{\text{joint}} \simeq \sum_k (\Theta-\Theta_*)^\top \mathcal{I}_k(\Theta-\Theta_*) = (\Theta-\Theta_*)^\top\left(\sum_k \mathcal{I}_k\right)(\Theta-\Theta_*). \tag{43}$$

Hence, the combined precision with which GR can be falsified is governed by the total information content

$$\mathcal{I}_{\text{tot}} = \sum_k \mathcal{I}_k = \sum_k \Sigma_k^{-1}, \tag{44}$$

demonstrating that the KTC aggregates the constraining power of all sectors while penalizing mutual inconsistency. Because the covariances of the three sectors are nearly orthogonal in orientation, the determinant of $\mathcal{I}_{\text{tot}}$ is greatly enhanced relative to any single dataset, yielding a sharp and highly localized joint posterior for (M, a). The scaling of T^2 with experimental uncertainties can be illustrated in a simple limiting case: if the dominant inconsistency is a bias δM in one sector's inferred mass while the others are unbiased and the mass uncertainty in that sector is σ_M, then the contribution to the statistic scales as

$$T^2 \sim \left(\frac{\delta M}{\sigma_M}\right)^2, \tag{45}$$

with analogous behavior for a and for multi-parameter offsets through the full quadratic form $\delta\Theta^\top\Sigma^{-1}\delta\Theta$. This shows that even sub-percent inconsistencies among sectors can yield a measurable excess in T^2 when the sectoral uncertainties are sufficiently small. The closure test is thus extremely sensitive to departures from Kerr consistency, with the signal-to-noise ratio of any deviation scaling linearly with the combined precision of the sectoral estimates.

V. INTERPRETATION OF FAILURES

The diagnostic power of the Kerr Trisector Closure resides not only in detecting a violation of consistency but also in identifying *which physical sector* is responsible for the discrepancy. When a statistically significant excess of the closure statistic T^2 is measured, the pattern of residuals among the three sectors provides information about the underlying nature of the deviation. Let the total residual vector introduced in Eq. (32) be decomposed into its sectoral components

$$\mathbf{r} = \begin{bmatrix}\delta\Theta_{\text{insp}}\\ \delta\Theta_{\text{ring}}\\ \delta\Theta_{\text{img}}\end{bmatrix}, \qquad \delta\Theta_k = \widehat{\Theta}_k - \bar{\Theta}, \tag{46}$$

where each $\delta\Theta_k \in \mathbb{R}^2$ represents the deviation of sector k from the global best-fit Kerr parameters $\bar{\Theta}$.

A. Sectoral Contributions to the Closure Statistic

The total statistic can be written as a sum of sectoral contributions weighted by the inverse covariances:

$$T^2 = \sum_k (\delta\Theta_k)^\top \Sigma_k^{-1}\delta\Theta_k. \tag{47}$$

Each term in Eq. (47) quantifies the contribution of one observational domain to the total deviation from metric closure. By defining the partial statistic

$$T_k^2 = (\delta\Theta_k)^\top\Sigma_k^{-1}\delta\Theta_k, \tag{48}$$

the relative weight of each domain in the total discrepancy can be expressed through normalized components

$$\eta_k = \frac{T_k^2}{T^2}, \qquad \sum_k \eta_k = 1. \tag{49}$$

The set $\{\eta_{\text{insp}}, \eta_{\text{ring}}, \eta_{\text{img}}\}$ thus forms a statistical signature that identifies which physical regime dominates the observed inconsistency. Values of η_k close to unity indicate that sector as the primary source of deviation, while comparable contributions among several sectors suggest a more global breakdown of the Kerr description.

B. Classification of Failure Modes

The distinct physical interpretations of the deviation can be summarized through the limiting configurations of η_k.

a. Perturbative-sector deviations. If T^2 is dominated by T^2_{ring} ($\eta_{\text{ring}} \approx 1$), the inconsistency arises in the linear-perturbation domain near the horizon. Such a pattern implies that the inspiral dynamics and photon-ring geometry remain well described by the Kerr metric, whereas the quasinormal-mode spectrum exhibits anomalies. Possible physical origins include modifications of the near-horizon boundary conditions, as predicted by models of horizon-scale structure, gravitational-wave *echoes*, or the presence of exotic compact objects lacking true event horizons. Mathematically, these scenarios correspond to deformations of the effective potential in the Teukolsky equation,

$$\mathcal{T}[\psi_4] \to \mathcal{T}[\psi_4] + \delta V(r)\psi_4, \tag{50}$$

where $\delta V(r)$ represents the perturbation induced by new physics or non-Kerr geometry.

b. Optical-sector deviations. If the dominant contribution arises from the imaging sector ($\eta_{\text{img}} \approx 1$), the discrepancy is localized in the null-geodesic structure of spacetime. Such deviations may originate from modified photon propagation, dispersive or birefringent effects, or the presence of non-vacuum stress–energy distributions in the near-horizon region. These effects can be modeled at leading order by an effective metric deformation

$$g^{\text{eff}}_{\mu\nu} = g^{\text{Kerr}}_{\mu\nu} + \epsilon\, h^{(\gamma)}_{\mu\nu}, \tag{51}$$

where $h^{(\gamma)}_{\mu\nu}$ encodes corrections that couple preferentially to null trajectories (e.g., via quantum or plasma-induced refractive indices). In this case, the gravitational dynamics may remain consistent with GR while the optical sector reflects new interaction channels between spacetime curvature and radiation.

c. Dynamical-sector deviations. If the inspiral term dominates ($\eta_{\rm insp} \approx 1$), the inconsistency originates from deviations in the conservative or dissipative dynamics of the binary system. Potential explanations include long-range scalar or vector fields leading to dipole radiation, violations of the strong equivalence principle, or modifications of the effective two-body potential at post-Newtonian order. Formally, these effects can be represented as an additional potential term in the orbital Hamiltonian,

$$H_{\rm eff} = H_{\rm Kerr}(M, a) + \delta H_{\rm ext}(r, p; \lambda), \tag{52}$$

where λ parameterizes the coupling strength of the beyond-GR interaction. Such deviations would primarily affect the phase evolution of gravitational waves during inspiral while leaving the ringdown and imaging observables largely unaltered.

d. (iv) Multi-sector or global deviations. A non-negligible contribution from all three sectors ($\eta_{\rm insp} \sim \eta_{\rm ring} \sim \eta_{\rm img}$) indicates a breakdown of the universality of the metric itself. This scenario cannot be attributed to a single observational channel but rather implies that the Kerr family of solutions fails to represent the underlying geometry across all regimes simultaneously. Mathematically, this would correspond to a deformation of the Einstein field equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu} + \mathcal{F}_{\mu\nu}, \tag{53}$$

where the tensor $\mathcal{F}_{\mu\nu} \neq 0$ encodes additional geometric or quantum contributions not captured by classical GR. A confirmed detection of such a global failure would constitute evidence for physics beyond the Einsteinian framework.

C. Practical Diagnostic Procedure

In practice, the evaluation of the partial contributions T^2_k provides a quantitative basis for diagnosing the failure mode. Posterior sampling in the combined parameter space allows one to compute the mean residual vectors $\langle \delta\Theta_k \rangle$ and to test their orthogonality or alignment across sectors. If the residuals are aligned along a common direction, the deviation likely reflects a correlated systematic bias (for instance, an incorrect distance calibration). If they are mutually orthogonal, the discrepancy is more likely of physical origin. Thus, the decomposition of T^2 into its sectoral components not only determines whether GR fails but also reveals *how* and *where* the failure occurs within the layered structure of spacetime phenomenology.

VI. IMPLEMENTATION AND PROSPECTS

Realizing the Kerr Trisector Closure (KTC) as an empirical test requires coordinated observations that access all three physical regimes (dynamical, perturbative, and geometrical–optical) for a single astrophysical system. In practice, this entails combining multi-band gravitational-wave (GW) measurements of compact-binary coalescences with very-long-baseline interferometric (VLBI) imaging of the merger remnant or its environment, together with electromagnetic (EM) observations that identify the host galaxy and constrain the distance scale. The essential requirement is that the same astrophysical object be independently resolved by all three sectors so that the resulting parameter estimates $\widehat{\Theta}_k$ correspond to a common spacetime.

A. Multi-band Gravitational-wave Observations

GW astronomy gives us a natural way to access the dynamic and perturbative sectors of the KTC. Space-based detectors like LISA can see a single binary black hole years before it merges, and ground-based interferometers like the Einstein Telescope (ET) and Cosmic Explorer (CE) can see it during the last seconds of inspiral, merger, and ringdown. The observation from space limits the inspiral phase with great precision, allowing for an exact calculation of the chirp mass and the effective spin parameter based on the early-time phasing of the waveform. The ground-based detection then separates the nonlinear merger from the quasinormal-mode ringdown, allowing M and a to be extracted independently from the perturbative sector.

Formally, the inspiral likelihood of Eq. (21) and the ringdown likelihood of Eq. (24) become simultaneously measurable for the same source, with correlated priors propagated through Bayesian inference pipelines. The joint posterior $p(\Theta|d_{\rm insp}, d_{\rm ring})$ then furnishes both $\widehat{\Theta}_{\rm insp}$ and $\widehat{\Theta}_{\rm ring}$, as well as their cross-covariance, permitting a direct evaluation of the intermediate contribution to T^2.

The forecasted precision of multi-band measurements has been estimated in several studies. For a typical $30 + 30\, M_\odot$ binary observed by LISA and ET, fractional uncertainties can reach

$$\frac{\sigma_M}{M} \lesssim 10^{-3}, \qquad \sigma_a \lesssim 10^{-2}, \tag{54}$$

for the inspiral parameters, and comparable accuracy for the dominant $(l, m, n) = (2, 2, 0)$ ringdown mode. Such precision is sufficient to detect sub-percent inconsistencies between the inspiral and ringdown sectors if present.

B. Electromagnetic Counterparts and Distance Calibration

An electromagnetic counterpart to the merger event, whether in the form of an optical transient, a jet, or a host-galaxy association, plays a crucial role in providing the redshift z and the luminosity distance D. These quantities link the mass M measured in gravitational units to its apparent angular scale in the imaging domain. The inclusion of EM information converts the imaging measurement of the shadow angular diameter $\theta_{\rm sh} = d_{\rm sh}(\Theta, i)/D$ into a direct constraint on the physical mass, thereby anchoring the geometrical–optical sector to the same physical parameters as the GW domains. Simultaneous GW–EM observations of supermassive black-hole binaries, particularly those expected in gas-rich galactic mergers, provide promising candidates for future KTC realizations.

C. Very-long-baseline Interferometry of the Remnant

The optical sector of the KTC requires horizon-scale imaging of the same black hole whose inspiral and ringdown are observed in GWs. This is achievable through next-generation VLBI facilities such as the Event Horizon Telescope (EHT) and its planned successor, the ngEHT, which will offer improved sensitivity and dynamic range. For nearby supermassive systems ($z \lesssim 0.05$) the EHT resolution of tens of microarcseconds allows the direct measurement of the shadow diameter and sub-ring structure, yielding the parameters $\widehat{M}_{\rm img}$ and $\widehat{a}_{\rm img}$. In the case of stellar-mass binaries, direct imaging of the remnant horizon remains infeasible, but indirect optical analogues, such as X-ray reflection spectroscopy and relativistic iron-line measurements, can provide equivalent constraints on the photon-orbit structure. The imaging likelihood of Eq. (26) can then be evaluated using a measured shadow size estimator (e.g., $\widehat{\theta}_{\rm sh}$), marginalizing over the inclination and distance priors supplied by EM data. Combining this with the GW-derived posteriors completes the three-sector dataset necessary for computing the closure statistic T^2.

D. Feasibility and Forecasts

To assess observational feasibility, one may model the expected signal-to-noise ratios (SNRs) in each domain and propagate them into parameter covariances using Fisher-matrix approximations. If the uncertainties in each sector scale as $\Sigma_k \propto {\rm SNR}_k^{-2}$, the overall precision of the KTC is governed by the combined Fisher information

$$\mathcal{I}_{\rm tot} = \sum_k \Sigma_k^{-1} \propto \sum_k {\rm SNR}_k^2. \tag{55}$$

For a system with moderate redshift $z \simeq 0.05$ observed by LISA (${\rm SNR}_{\rm insp} \sim 100$), ET (${\rm SNR}_{\rm ring} \sim 200$), and ngEHT (${\rm SNR}_{\rm img} \sim 30$), the expected joint uncertainty in mass and spin would reach the level of

$$\frac{\sigma_M}{M} \sim 5 \times 10^{-4}, \qquad \sigma_a \sim 5 \times 10^{-3}, \tag{56}$$

corresponding to a sensitivity sufficient to detect deviations at the level of $\Delta M/M \sim 10^{-3}$ or $\Delta a \sim 10^{-2}$. Hence, any physical inconsistency larger than this threshold would produce a measurable excess in T^2 at the $> 5\sigma$ level.

E. Future Outlook

The coming decades will witness the confluence of observational technologies required to realize the KTC empirically. Space-based interferometers such as LISA will probe low-frequency inspirals of massive binaries, while third-generation ground-based detectors (ET and CE) will capture their mergers and ringdowns with unprecedented SNR. Concurrently, the ngEHT and millimeter-space-VLBI missions will provide angular resolutions capable of resolving event-horizon-scale features for a growing sample of nearby supermassive black holes. The synergy of these instruments will, for the first time, allow the simultaneous determination of the parameters (M, a) through three orthogonal channels, enabling a direct computation of the closure statistic and the first fully self-consistent test of the Kerr metric across dynamical, perturbative, and optical regimes.

VII. DISCUSSION

The Kerr Trisector Closure (KTC) provides a cross-domain consistency test for General Relativity based on a single underlying metric description. Rather than introducing sector-specific deformation parameters, the method evaluates whether one Kerr spacetime can simultaneously account for observables from orbital dynamics, ringdown relaxation, and photon propagation. Most existing tests focus on individual terms or regimes (e.g., post-Newtonian dynamics, waveform generation, or redshift observables), whereas the KTC imposes a joint consistency condition across these domains.

In GR, the same spacetime geometry governs orbital precession, the quasinormal-mode spectrum, and the deflection of null geodesics. The KTC formalism maps this shared dependence into a statistical test of metric universality. If the parameters (M, a) inferred from inspiral dynamics, ringdown spectroscopy, and photon-ring imaging are consistent within their covariances, the result supports a single Kerr description across the three sectors. If a statistically significant discrepancy is present, the magnitude and sectoral decomposition of T^2 identify the domain(s) contributing to the inconsistency without specifying an alternative theory. Traditional frameworks, including the parameterized post-Einsteinian (ppE) and parameterized post-Newtonian (PPN) formalisms, introduce explicit deformation parameters to represent deviations from GR[3, 12, 23]. The KTC instead treats closure among independently inferred (M, a) estimates as the observable diagnostic. In this approach, rejection of closure corresponds to a failure of a single-metric description to describe all three sectors for the same source. If closure is satisfied, the allowed parameter space for sector-dependent deviations is restricted by the joint uncertainties of the three estimates. If the closure statistic T^2 exceeds its null expectation, the inferred failure mode depends on the sectoral contributions: a perturbative-sector contribution indicates inconsistency in the quasinormal-mode inference, an optical-sector contribution indicates inconsistency in the null-geodesic inference, and comparable contributions across sectors indicate a more global inconsistency with the Kerr description.

The closure formalism can be generalized to other settings in which a stationary vacuum solution is characterized by a parameter vector. Extensions include charged black holes (Kerr–Newman), cosmological spacetimes parameterized by expansion and curvature, and scalar-tensor or vector-tensor theories in which different sectors couple differently to additional fields. In each case, the test evaluates consistency of parameter estimates inferred from distinct observables with a common spacetime model.

VIII. CONCLUSION

The Kerr Trisector Closure (KTC) provides a statistically defined approach for evaluating cross-domain consistency of General Relativity across distinct observational sectors. It requires that a single Kerr spacetime reproduce, within measurement uncertainty, the observables associated with orbital dynamics, perturbative relaxation, and null-geodesic structure, as quantified by the closure statistic T^2. Within the Gaussian framework adopted here, an excess of T^2 beyond its chi-squared expectation indicates inconsistency among sectoral parameter estimates, and the sectoral decomposition provides a diagnostic of which domain contributes to the discrepancy. This differs from parameterized-post-Einsteinian approaches based on assumed deformation templates by using disagreement among independently inferred (M, a) estimates as the observable indicator.

Upcoming multi-band gravitational-wave observations and horizon-scale imaging are expected to reach the precision required for an empirical implementation of the KTC. Interferometers in space and on the ground (LISA, ET, CE) constrain the dynamical and perturbative sectors, while next-generation VLBI arrays (ngEHT and related facilities) constrain the optical sector. Combining these measurements enables a direct evaluation of T^2 for astrophysical black holes as a metric-level self-consistency test in the strong-field regime. If closure is satisfied within statistical uncertainties, the results support a single Kerr description across the three sectors. If a statistically significant excess is observed, the test provides an empirical indication of sectoral inconsistency with the Kerr model without specifying an alternative theory.

Appendix A: Statistical derivation details

This appendix derives the generalized least-squares (GLS) estimator $\bar{\Theta}$, the projection matrix $\mathbf{P}$, and the null distribution of T^2. We follow the notation of Sec. IV.

1. Derivation of the GLS estimator

Consider the stacked measurement model

$$\widehat{\boldsymbol{\Theta}} = \mathbf{A}\,\Theta + \boldsymbol{\varepsilon}, \qquad \mathbb{E}[\boldsymbol{\varepsilon}] = 0, \qquad \mathrm{Cov}(\boldsymbol{\varepsilon}) = \mathbf{C}, \tag{A1}$$

where $\widehat{\boldsymbol{\Theta}} \in \mathbb{R}^6$, $\Theta \in \mathbb{R}^2$, and $\mathbf{C} \in \mathbb{R}^{6\times 6}$ is symmetric and positive definite.

The GLS estimate $\bar{\Theta}$ is defined as the minimizer of the weighted least-squares objective

$$J(\Theta) \equiv \left(\widehat{\boldsymbol{\Theta}} - \mathbf{A}\Theta\right)^{\top} \mathbf{C}^{-1} \left(\widehat{\boldsymbol{\Theta}} - \mathbf{A}\Theta\right). \tag{A2}$$

Expanding and differentiating with respect to Θ gives

$$J(\Theta) = \widehat{\boldsymbol{\Theta}}^{\top}\mathbf{C}^{-1}\widehat{\boldsymbol{\Theta}} - 2\,\Theta^{\top}\mathbf{A}^{\top}\mathbf{C}^{-1}\widehat{\boldsymbol{\Theta}} + \Theta^{\top}\mathbf{A}^{\top}\mathbf{C}^{-1}\mathbf{A}\,\Theta, \tag{A3}$$

$$\nabla_{\Theta} J(\Theta) = -2\,\mathbf{A}^{\top}\mathbf{C}^{-1}\widehat{\boldsymbol{\Theta}} + 2\,\mathbf{A}^{\top}\mathbf{C}^{-1}\mathbf{A}\,\Theta. \tag{A4}$$

Setting $\nabla_{\Theta} J(\Theta) = 0$ yields the normal equations

$$\mathbf{A}^{\top}\mathbf{C}^{-1}\mathbf{A}\,\bar{\Theta} = \mathbf{A}^{\top}\mathbf{C}^{-1}\widehat{\boldsymbol{\Theta}}. \tag{A5}$$

If $\mathbf{A}$ has full column rank, then $\mathbf{A}^{\top}\mathbf{C}^{-1}\mathbf{A}$ is invertible and

$$\bar{\Theta} = \left(\mathbf{A}^{\top}\mathbf{C}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{\top}\mathbf{C}^{-1}\widehat{\boldsymbol{\Theta}}, \tag{A6}$$

which matches Eq. (30).

2. Derivation of the projection matrix

Define the fitted value $\widehat{\boldsymbol{\Theta}}_{\mathrm{fit}} \equiv \mathbf{A}\bar{\Theta}$ and the residual $\mathbf{r} \equiv \widehat{\boldsymbol{\Theta}} - \widehat{\boldsymbol{\Theta}}_{\mathrm{fit}}$. Substituting the GLS estimator gives

$$\widehat{\boldsymbol{\Theta}}_{\mathrm{fit}} = \mathbf{A}\left(\mathbf{A}^{\top}\mathbf{C}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{\top}\mathbf{C}^{-1}\widehat{\boldsymbol{\Theta}}, \tag{A7}$$

$$\mathbf{r} = \left[\mathbb{I}_6 - \mathbf{A}\left(\mathbf{A}^{\top}\mathbf{C}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{\top}\mathbf{C}^{-1}\right]\widehat{\boldsymbol{\Theta}}. \tag{A8}$$

This motivates the definition

$$\mathbf{P} \equiv \mathbb{I}_6 - \mathbf{A}\left(\mathbf{A}^\top \mathbf{C}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^\top\mathbf{C}^{-1}, \tag{A9}$$

so that $\mathbf{r} = \mathbf{P}\widehat{\boldsymbol{\Theta}}$, consistent with Eq. (32).

The residual is orthogonal to the model space in the $\mathbf{C}^{-1}$ inner product. Using $\mathbf{A}^\top\mathbf{C}^{-1}\mathbf{A}\bar{\Theta} = \mathbf{A}^\top\mathbf{C}^{-1}\widehat{\boldsymbol{\Theta}}$,

$$\mathbf{A}^\top\mathbf{C}^{-1}\mathbf{r} = \mathbf{A}^\top\mathbf{C}^{-1}\widehat{\boldsymbol{\Theta}} - \mathbf{A}^\top\mathbf{C}^{-1}\mathbf{A}\bar{\Theta} = 0. \tag{A10}$$

This implies $\mathbf{PA} = 0$ and shows that $\mathbf{P}$ projects onto the $\mathbf{C}^{-1}$-orthogonal complement of col($\mathbf{A}$).

3. Null distribution of T^2

Under the null hypothesis (a common Kerr parameter pair Θ_*), assume

$$\widehat{\boldsymbol{\Theta}} \sim \mathcal{N}(\mathbf{A}\Theta_*, \mathbf{C}). \tag{A11}$$

Define the whitened residual

$$\mathbf{z} \equiv \mathbf{C}^{-1/2}\left(\widehat{\boldsymbol{\Theta}} - \mathbf{A}\Theta_*\right), \tag{A12}$$

where $\mathbf{C}^{-1/2}\mathbf{C}\mathbf{C}^{-1/2} = \mathbb{I}_6$. Then $\mathbf{z} \sim \mathcal{N}(0, \mathbb{I}_6)$.

The closure statistic is

$$T^2 = \mathbf{r}^\top\mathbf{C}^{-1}\mathbf{r} = \left(\mathbf{C}^{-1/2}\mathbf{r}\right)^\top\left(\mathbf{C}^{-1/2}\mathbf{r}\right), \tag{A13}$$

with $\mathbf{r} = \mathbf{P}\widehat{\boldsymbol{\Theta}}$. Under the null,

$$\mathbf{C}^{-1/2}\mathbf{r} = \mathbf{C}^{-1/2}\mathbf{P}\left(\mathbf{A}\Theta_* + \mathbf{C}^{1/2}\mathbf{z}\right) = \mathbf{C}^{-1/2}\mathbf{P}\mathbf{C}^{1/2}\mathbf{z}, \tag{A14}$$

since $\mathbf{PA} = 0$.

Define

$$\mathbf{M} \equiv \mathbf{C}^{-1/2}\mathbf{P}\mathbf{C}^{1/2}. \tag{A15}$$

Then $\mathbf{C}^{-1/2}\mathbf{r} = \mathbf{Mz}$ and

$$T^2 = \mathbf{z}^\top\mathbf{M}^\top\mathbf{M}\,\mathbf{z}. \tag{A16}$$

One can verify that $\mathbf{M}^\top\mathbf{M}$ is symmetric and idempotent, so it is an orthogonal projector. Its rank is

$$\nu = \text{rank}(\mathbf{P}) = 6 - \text{rank}(\mathbf{A}) = 4, \tag{A17}$$

which corresponds to $(K-1)p$ with $K = 3$ sectors and $p = 2$ parameters.

For $\mathbf{z} \sim \mathcal{N}(0, \mathbb{I}_6)$ and a rank-ν orthogonal projector $\mathbf{Q}$, the quadratic form $\mathbf{z}^\top\mathbf{Q}\mathbf{z}$ is χ^2_ν-distributed. Applying this with $\mathbf{Q} = \mathbf{M}^\top\mathbf{M}$ yields

$$T^2 \sim \chi^2_\nu, \qquad \nu = 4, \tag{A18}$$

which reproduces the null distribution stated in Sec. IV.

Appendix B: Example covariance and T^2 evaluation

This appendix gives a simple numerical example of the KTC calculation. The goal is to show how to assemble $\widehat{\boldsymbol{\Theta}}$ and $\mathbf{C}$, compute $\bar{\Theta}$, $\mathbf{r}$, and T^2, and convert T^2 into a p-value.

1. Toy sector estimates and covariances

Assume three independent sector estimates of $\Theta = (M, a)$,

$$\widehat{\Theta}_{\text{insp}} = (10.00,\ 0.60), \quad \Sigma_{\text{insp}} = \begin{pmatrix} 0.30^2 & 0 \\ 0 & 0.05^2 \end{pmatrix}, \tag{B1}$$

$$\widehat{\Theta}_{\text{ring}} = (10.20,\ 0.55), \quad \Sigma_{\text{ring}} = \begin{pmatrix} 0.40^2 & 0 \\ 0 & 0.06^2 \end{pmatrix}, \tag{B2}$$

$$\widehat{\Theta}_{\text{img}} = (9.80,\ 0.65), \quad \Sigma_{\text{img}} = \begin{pmatrix} 0.50^2 & 0 \\ 0 & 0.07^2 \end{pmatrix}. \tag{B3}$$

We take the full covariance to be block diagonal,

$$\mathbf{C} = \text{blockdiag}\left(\Sigma_{\text{insp}}, \Sigma_{\text{ring}}, \Sigma_{\text{img}}\right), \tag{B4}$$

so that Eq. (47) applies.

2. Compute the GLS mean and residuals

Because the covariances are diagonal and independent, the GLS mean $\bar{\Theta} = (\bar{M}, \bar{a})$ reduces to component-wise inverse-variance weighting. For the mass,

$$\bar{M} = \frac{10.00/0.30^2 + 10.20/0.40^2 + 9.80/0.50^2}{1/0.30^2 + 1/0.40^2 + 1/0.50^2} \simeq 10.02. \tag{B5}$$

For the spin,

$$\bar{a} = \frac{0.60/0.05^2 + 0.55/0.06^2 + 0.65/0.07^2}{1/0.05^2 + 1/0.06^2 + 1/0.07^2} \simeq 0.596. \tag{B6}$$

The sector residuals $\delta\Theta_k = \widehat{\Theta}_k - \bar{\Theta}$ are then

$$\delta\Theta_{\text{insp}} \simeq (-0.02,\ 0.004), \tag{B7}$$

$$\delta\Theta_{\text{ring}} \simeq (-0.18,\ -0.046), \tag{B8}$$

$$\delta\Theta_{\text{img}} \simeq (-0.22,\ 0.054). \tag{B9}$$

3. Compute T^2 and a p-value

With independent sectors,

$$T^2 = \sum_k (\delta\Theta_k)^\top \Sigma_k^{-1} \delta\Theta_k. \tag{B10}$$

For the values above, the three contributions are

$$T^2_{\rm insp} \simeq \frac{(-0.02)^2}{0.30^2} + \frac{(0.004)^2}{0.05^2} \simeq 0.011, \tag{B11}$$

$$T^2_{\rm ring} \simeq \frac{(0.18)^2}{0.40^2} + \frac{(-0.046)^2}{0.06^2} \simeq 0.790, \tag{B12}$$

$$T^2_{\rm img} \simeq \frac{(-0.22)^2}{0.50^2} + \frac{(0.054)^2}{0.07^2} \simeq 0.789, \tag{B13}$$

so

$$T^2 \simeq 1.59. \tag{B14}$$

Under the null, $T^2 \sim \chi^2_\nu$ with $\nu = 4$. For $\nu = 4$, the survival function has the closed form

$$p \equiv \mathbb{P}(\chi^2_4 \geq T^2) = \exp\left(-\frac{T^2}{2}\right)\left(1 + \frac{T^2}{2}\right). \tag{B15}$$

Using $T^2 \simeq 1.59$ gives $p \simeq 0.81$.

Appendix C: Shadow and ringdown reference formulas

This appendix summarizes practical formulas used in the imaging and ringdown sectors. The goal is to make the numerical evaluation of $d_{\rm sh}(M, a, i)$ and the use of tabulated quasinormal-mode (QNM) frequencies explicit.

1. Numerical evaluation of the shadow diameter

Section III defines the shadow boundary through the unstable spherical photon orbits parameterized by r, with $\xi(r)$ and $\eta(r)$ given by Eq. (18) and mapped to image-plane coordinates (α, β) through Eq. (17). A direct numerical evaluation of the horizontal shadow diameter

$$d_{\rm sh}(M, a, i) = \alpha_{\rm max}(M, a, i) - \alpha_{\rm min}(M, a, i) \tag{C1}$$

can be implemented as follows.

1. Fix (M, a) and inclination i. Work in geometric units $G = c = 1$.

2. Choose a grid in r covering the range of unstable spherical photon orbits. In practice one can scan a broad interval, for example $r \in [r_+ + \epsilon,\, r_{\rm max}]$ with $\epsilon > 0$, and then keep only points that satisfy the admissibility conditions below.

3. For each grid point r, compute $\xi(r)$ and $\eta(r)$ from Eq. (18).

4. Map (ξ, η) to (α, β) using Eq. (17). For a given (ξ, η), the two signs in β correspond to the upper and lower halves of the shadow and do not affect α.

5. Impose admissibility conditions. A simple operational choice is to keep only points for which β is real, equivalently

$$\eta + a^2 \cos^2 i - \xi^2 \cot^2 i \geq 0. \tag{C2}$$

6. From the retained points, compute

$$\alpha_{\rm min} = \min\{\alpha(r)\}, \qquad \alpha_{\rm max} = \max\{\alpha(r)\}, \tag{C3}$$

and then evaluate $d_{\rm sh} = \alpha_{\rm max} - \alpha_{\rm min}$.

The angular diameter is $\theta_{\rm sh} = d_{\rm sh}/D$ for a source at distance D. In parameter inference, i and D can be treated as nuisance parameters and marginalized as described in Sec. IV.

2. Dimensionless QNM frequencies

The ringdown scaling relation used in Sec. V is

$$\omega_{lmn}(M, a) \simeq \frac{1}{M}\,\tilde{\omega}_{lmn}(\chi), \qquad \chi \equiv a/M, \tag{C4}$$

where $\tilde{\omega}_{lmn}(\chi)$ is a dimensionless complex frequency that depends on χ and on the mode indices (l, m, n). Tabulations and fitting formulas for $\tilde{\omega}_{lmn}(\chi)$ are available in the literature (see Ref. [10]).

Table I lists representative values for the non spinning case $\chi = 0$ (Schwarzschild). These values are useful as a simple check of units and conventions.

TABLE I. Representative dimensionless QNM frequencies $\tilde{\omega}_{lmn}(\chi = 0)$ for Schwarzschild, written as $\tilde{\omega} = \tilde{\omega}^{(r)} + i\tilde{\omega}^{(i)}$. The damping time is $\tau = 1/|\omega^{(i)}|$. Values are taken from standard tabulations (see Ref. [10]).

l	m	n	$\tilde{\omega}_{lmn}(\chi = 0)$
2	2	0	$0.37367 - 0.08896\,i$
2	2	1	$0.34671 - 0.27391\,i$
3	3	0	$0.59944 - 0.09270\,i$

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